On the Eigenfunction Expansion of Electromagnetic Dyadic Green's Functions in Rectangular Cavities and Waveguides

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Abstract—**The electric dyadic Green's functions of both the first and the second kinds due to the presence of electric and equivalent magnetic sources in rectangular cavities are obtained. A method for directly reducing the dyadic Green's functions for a rectangular cavity to those for a semi-infinite and an infinite rectangular waveguides is presented. The Green dyads of the second kind for an infinite and a semi-infinite rectangular waveguides, and a rectangular cavity are obtained.**

I. Introduction

Early in 1960, Collin published the $\hat{\boldsymbol{y}}\hat{\boldsymbol{y}}$-component of the electric Green dyad of the first kind for a semi-infinite rectangular waveguide [1]. In 1971, Tai [2] introduced the magnetic dyadic Green's function of the first kind for an infinite rectangular waveguide in terms of eigenfunction expansion. Rahmat-Samii [4] in 1975 and Tai [3] in 1976 discussed in detail the magnetic Green dyads of the first kind in rectangular cavities and infinite waveguides. Recently, a more detailed treatment of the three-dimensional Green dyad for rectangular waveguides and cavities under rectangular coordinates has been given by Balanis [7], Collin [1] and Tai [2].

So far, the Green dyads of the first kind in rectangular waveguides and cavities have been well-investigated. However, the Green dyads of the second kind in a semi-infinite rectangular waveguide have not been given although the Green dyads of the first kind have been presented recently by Tai [2]. In practical problems, it is sometimes necessary to calculate the electromagnetic fields due to both electric and equivalent magnetic current sources. Consequently, the Green dyads of the second kind are needed. The $\hat{\boldsymbol{y}}\hat{\boldsymbol{x}}$- and $\hat{\boldsymbol{y}}\hat{\boldsymbol{z}}$-components of the electric Green dyads of the second kind were presented by Jarem [6] in 1987 for a semi-infinite waveguide and presented by Liang *et al.* [8] in 1992 for a rectangular cavity.

This paper defines the electromagnetic fields in terms of Green dyads when *both* the electric and the equivalent magnetic sources exist in a rectangular semi-infinite or infinite waveguide and cavity. The electric dyadic Green's functions of the first and second kinds for a rectangular waveguide and a rectangular cavity are presented. Besides, a method for reducing the Green dyads in a rectangular cavity to those in a semi-infinite and an infinite waveguides is presented. With this method, it becomes very easy and direct to obtain the Green dyads in a rectangular waveguide with multiple loads when the Green dyad in a rectangular cavity filled with multi-layered medium has been derived. The magnetic Green dyads of the second kind derived here is compared with those reported in the literature and the corresponding correctness of the solutions discussed.

II. Fundamental Problem

The electromagnetic radiation fields, $\boldsymbol{E}$ and $\boldsymbol{H}$ in a rectangular cavity or waveguide, contributed by the electric current distributions $\boldsymbol{J}$ and $\boldsymbol{M}$ located in the rectangular cavity or waveguide may be expressed in terms of the integrals of the *electric* and *magnetic* Green dyads of the first and second kinds from

$$\boldsymbol{E}(\boldsymbol{r}) = -j\omega\mu \iiint_V \overline{\boldsymbol{G}}_{e1}(\boldsymbol{r},\boldsymbol{r}')\cdot\boldsymbol{J}(\boldsymbol{r}')\,dV' - \iiint_V \nabla\times\overline{\boldsymbol{G}}_{e2}(\boldsymbol{r},\boldsymbol{r}')\cdot\boldsymbol{M}(\boldsymbol{r}')\,dV', \tag{1a}$$

$$\boldsymbol{H}(\boldsymbol{r}) = \iiint_V \nabla\times\overline{\boldsymbol{G}}_{e1}(\boldsymbol{r},\boldsymbol{r}')\cdot\boldsymbol{J}(\boldsymbol{r}')\,dV' - j\omega\varepsilon \iiint_V \overline{\boldsymbol{G}}_{e2}(\boldsymbol{r},\boldsymbol{r}')\cdot\boldsymbol{M}(\boldsymbol{r}')\,dV', \tag{1b}$$

where ε, μ and σ stand for the permittivity, permeability and conductivity of the medium, respectively; V identifies the volume occupied by the sources; and the subscripts $e1$ and $e2$ denote the electric Green dyads of the first and second kinds. A time dependence $j\omega t$ is assumed for the fields throughout the paper.

Due to the presence of both an electric and an equivalent magnetic current sources, the boundary conditions on the walls of a conducting cavity can be written as

$$\hat{n}\times\boldsymbol{E} = -\boldsymbol{K}_M, \quad \hat{n}\cdot\boldsymbol{D} = \rho_J; \tag{2a}$$

$$\hat{n}\times\boldsymbol{H} = \boldsymbol{K}_J, \quad \hat{n}\cdot\boldsymbol{B} = \rho_M; \tag{2b}$$

where $\boldsymbol{K}_{J,M}$ and $\rho_{J,M}$ denote the surface current and charge densities due to the electromagnetic sources $\boldsymbol{J}$ and $\boldsymbol{M}$ with arbitrary distributions, respectively. For an electrically perfectly conducting cavity, the boundary conditions of electromagnetic Green dyads in (2) can be given by:

$$\hat{n}\times\overline{\boldsymbol{G}}_{e1}(\boldsymbol{r},\boldsymbol{r}') = 0, \quad \hat{n}\cdot\nabla\times\overline{\boldsymbol{G}}_{e1}(\boldsymbol{r},\boldsymbol{r}') = 0, \tag{3a}$$

$$\hat{n}\times\nabla\times\overline{\boldsymbol{G}}_{e2}(\boldsymbol{r},\boldsymbol{r}') = 0, \quad \hat{n}\cdot\overline{\boldsymbol{G}}_{e2}(\boldsymbol{r},\boldsymbol{r}') = 0. \tag{3b}$$

For a magnetically perfectly conducting cavity, the boundary conditions of Green dyads are duals of (3a), (3b), therefore the corresponding formulae should be duals of what we derived here.

According to the method of scattering superposition, the electric dyadic Green's functions of the first and second kinds $\overline{\boldsymbol{G}}_{\substack{e1\\e2}}(\boldsymbol{r},\boldsymbol{r}')$ can be considered as the sum of the unbounded (with respect to z-direction) dyad $\overline{\boldsymbol{G}}_{\substack{e1\\e2}0}(\boldsymbol{r},\boldsymbol{r}')$ and the scattering Green dyad $\overline{\boldsymbol{G}}_{\substack{e1\\e2}s}(\boldsymbol{r},\boldsymbol{r}')$ contributed by the interfaces perpendicular to the z-direction, that is,

$$\overline{\boldsymbol{G}}_{\substack{e1\\e2}}(\boldsymbol{r},\boldsymbol{r}') = \overline{\boldsymbol{G}}_{\substack{e1\\e2}0}(\boldsymbol{r},\boldsymbol{r}') + \overline{\boldsymbol{G}}_{\substack{e1\\e2}s}(\boldsymbol{r}\boldsymbol{r}'). \tag{4}$$

where the unbounded Green dyad, $\overline{\boldsymbol{G}}_{\substack{e1\\e2}0}$, consisting of singularity and principal value contributions is given as follows:

$$\overline{\boldsymbol{G}}_{\substack{e1\\e2}0}(\boldsymbol{r},\boldsymbol{r}') = -\frac{\hat{z}\hat{z}\delta(\boldsymbol{r}-\boldsymbol{r}')}{k^2} - \frac{j}{ab}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{2-\delta_0}{\gamma k_c^2} \cdot\left[\boldsymbol{M}_{\substack{e\\o}mn}(\mp\gamma)\boldsymbol{M}'_{\substack{e\\o}mn}(\pm\gamma) + \boldsymbol{N}_{\substack{o\\e}mn}(\mp\gamma)\boldsymbol{N}'_{\substack{o\\e}mn}(\pm\gamma)\right], z \gtrless z' \quad (-\infty < z, z' < \infty), \tag{5}$$

where the rectangular vector wave functions are given in the 1st edition of Tai's book [2], δ_0 (= 1 for m or n = 0, and 0 otherwise) denotes the Kronecker delta, $\gamma^2 = k^2 - k_c^2 = k^2 - (n\pi/a)^2 - (m\pi/b)^2$, and k is designated as $k = \omega\sqrt{\mu\varepsilon(1-(j\sigma/\omega\varepsilon))}$.

It is noticed that the electric Green dyads of the first and second kinds have a singularity contributed by the source in the source region. However, the magnetic Green dyads of the first and second

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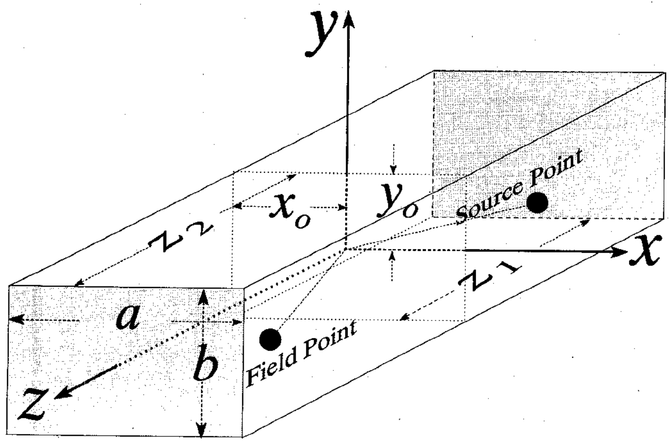

Fig. 1. Geometry of the rectangular cavity.

kinds do not have because the singularity term is cancelled by the derivatives of the delta function and the unit step function at the source point.

III. Rectangular Cavity

The geometry of the rectangular cavity is shown in Fig. 1. The scattering dyadic Green's function represents the contribution due to the presence of the cavity interfaces perpendicular to z-direction. Thus, taking the reflected waves into account, we may construct the scattering Green dyad accordingly as follows:

$$\begin{aligned}\overline{\boldsymbol{G}}_{\genfrac{}{}{0pt}{}{e1}{e2}s}(\boldsymbol{r},\boldsymbol{r}') = &-\frac{j}{ab}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{2-\delta_0}{\gamma k_c^2}\\ &\cdot\{\boldsymbol{M}_{\genfrac{}{}{0pt}{}{e}{o}mn}(\gamma)[\alpha^{M}_{\genfrac{}{}{0pt}{}{e}{o}mn}\boldsymbol{M}'_{\genfrac{}{}{0pt}{}{e}{o}mn}(\gamma)+\beta^{M}_{\genfrac{}{}{0pt}{}{e}{o}mn}\boldsymbol{M}'_{\genfrac{}{}{0pt}{}{e}{o}mn}(-\gamma)]\\ &+\boldsymbol{N}_{\genfrac{}{}{0pt}{}{o}{e}mn}(\gamma)[\alpha^{N}_{\genfrac{}{}{0pt}{}{o}{e}mn}\boldsymbol{N}'_{\genfrac{}{}{0pt}{}{o}{e}mn}(\gamma)+\beta^{N}_{\genfrac{}{}{0pt}{}{o}{e}mn}\boldsymbol{N}'_{\genfrac{}{}{0pt}{}{o}{e}mn}(-\gamma)]\\ &+\boldsymbol{M}_{\genfrac{}{}{0pt}{}{e}{o}mn}(-\gamma)[\alpha'^{M}_{\genfrac{}{}{0pt}{}{e}{o}mn}\boldsymbol{M}'_{\genfrac{}{}{0pt}{}{e}{o}mn}(\gamma)+\beta'^{M}_{\genfrac{}{}{0pt}{}{e}{o}mn}\boldsymbol{M}'_{\genfrac{}{}{0pt}{}{e}{o}mn}(-\gamma)]\\ &+\boldsymbol{N}_{\genfrac{}{}{0pt}{}{o}{e}mn}(-\gamma)[\alpha'^{N}_{\genfrac{}{}{0pt}{}{o}{e}mn}\boldsymbol{N}'_{\genfrac{}{}{0pt}{}{o}{e}mn}(\gamma)+\beta'^{N}_{\genfrac{}{}{0pt}{}{o}{e}mn}\boldsymbol{N}'_{\genfrac{}{}{0pt}{}{o}{e}mn}(-\gamma)]\}\\ &(z_1 < z, z' < z_2) \qquad (6)\end{aligned}$$

where the coefficients are determined from the boundary conditions and given below

$$\alpha^{M,N}_{\genfrac{}{}{0pt}{}{e}{o}mn} = (\mp)(+-)\frac{\exp^{-j\gamma(z_1+z_2)}}{2j\sin[\gamma(z_2-z_1)]}, \qquad (7a)$$

$$\alpha'^{M,N}_{\genfrac{}{}{0pt}{}{e}{o}mn} = \frac{\exp^{-j\gamma(z_2-z_1)}}{2j\sin[\gamma(z_2-z_1)]}, \qquad (7b)$$

$$\beta^{M,N}_{\genfrac{}{}{0pt}{}{e}{o}mn} = \frac{\exp^{-j\gamma(z_2-z_1)}}{2j\sin[\gamma(z_2-z_1)]}, \qquad (7c)$$

$$\beta'^{M,N}_{\genfrac{}{}{0pt}{}{e}{o}mn} = (\mp)(+-)\frac{\exp^{j\gamma(z_1+z_2)}}{2j\sin[\gamma(z_2-z_1)]}, \qquad (7d)$$

and the upper-lower and left-right notation of $(\mp)(+-)$ is designated for the subscript and superscript $(\genfrac{}{}{0pt}{}{e1}{e2})(MN)$.

So far, we derived both the electric Green dyads of the first and second kinds when an electric and an equivalent magnetic current distributions are present in a rectangular cavity simultaneously. It is found that the expressions of the electric Green dyads of the first kind in (4) after substitution of (5) and (6) for a rectangular cavity has exactly the same form of the dyadic Green's function in (33) presented by Tai [3], while the second kind is presented in the paper for the first time.

Presently, the $\hat{\boldsymbol{y}}\hat{\boldsymbol{y}}$-component of the electric Green dyads of the first and second kinds for a rectangular cavity has been given by Liang *et al.* In [8], the $\hat{\boldsymbol{y}}\hat{\boldsymbol{y}}$-component of the electric dyadic Green's function of the first kind can be found from

$$\left(-j\omega+\frac{\partial^2}{j\omega\mu\epsilon\partial y^2}\right)G_{Ayy}(x,y,z;x',y',z').$$

It can be seen that the $\hat{\boldsymbol{y}}\hat{\boldsymbol{y}}$-component of the dyadic Green function given here has exactly the same form as that given by Liang *et al.* [8] by letting $k=k_0$, $k_m^2=(m\pi/b)^2-k^2$ and $\Gamma_{nm}^2=(n\pi/a)^2+k_m^2=-\gamma^2$. Meanwhile, an agreement is also found between the dyadic Green's functions of the second kind obtained in this paper and those derived partially by Liang *et al.* [8]. However, it should be pointed out that the components $(\partial/\partial z)G_{Fxx}$ and $(\partial/\partial x)G_{Fzz}$ presented by Liang *et al.* [8] for the vector potential correspond to the components $\hat{\boldsymbol{y}}\cdot\nabla\times\overline{\boldsymbol{G}}_{e2}\cdot\hat{\boldsymbol{x}}$ and $\hat{\boldsymbol{y}}\cdot\nabla\times\overline{\boldsymbol{G}}_{e2}\cdot\hat{\boldsymbol{z}}$ given here for the EM fields are not the same. It is very easy to confuse those components G_{Fxx} and G_{Fzz} presented there with these components G_{E_2xx} and G_{E_2zz}.

IV. Simple Reduction of Green Dyads in a Cavity

A. Semi-Infinite Rectangular Waveguide

Although the electric Green dyads of the first kind for a semi-infinite rectangular waveguide have been obtained [2], the Green dyads of the second kind for a semi-infinite rectangular waveguide has not been presented in the literature, to authors' knowledge. Applying the boundary conditions, we can find these dyads. However, an efficient and simple method can be used to reduce the Green dyads for a rectangular cavity to those for a semi-infinite and an infinite rectangular waveguides, respectively.

Applying the Sommefeld radiation condition, we find that the terms containing the parameters $\alpha^{M,N}_{\genfrac{}{}{0pt}{}{e}{o}mn}$ and $\beta^{M,N}_{\genfrac{}{}{0pt}{}{e}{o}mn}$ vanish as $z_2\to\infty$. The coefficient, $\beta'^{M,N}_{\genfrac{}{}{0pt}{}{e}{o}mn}$, is given as follows

$$\beta'^{M,N}_{\genfrac{}{}{0pt}{}{e}{o}mn} = (\mp)(+-)e^{j2\gamma z_1}, \qquad (8)$$

while the coefficient, $\alpha'^{M,N}_{\genfrac{}{}{0pt}{}{e}{o}mn}$, vanishes. In fact, the coefficients in the case of rectangular semi-infinite waveguide can be reduced easily from (7a) and (7d) if we let $e^{-j\gamma z_2}\to 0$[1] as $z_2\to\infty$. Thus, the dyadic Green's function in (6) may be reduced to the following Green dyad

$$\begin{aligned}\overline{\boldsymbol{G}}_{\genfrac{}{}{0pt}{}{e1}{e2}s}(\boldsymbol{r},\boldsymbol{r}') = &-\frac{j}{ab}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{2-\delta_0}{\gamma k_c^2}\\ &\cdot[\beta'^{M}_{\genfrac{}{}{0pt}{}{e}{o}mn}\boldsymbol{M}_{\genfrac{}{}{0pt}{}{e}{o}mn}(-\gamma)\boldsymbol{M}'_{\genfrac{}{}{0pt}{}{e}{o}mn}(-\gamma)\\ &+\beta'^{N}_{\genfrac{}{}{0pt}{}{o}{e}mn}\boldsymbol{N}_{\genfrac{}{}{0pt}{}{o}{e}mn}(-\gamma)\boldsymbol{N}'_{\genfrac{}{}{0pt}{}{o}{e}mn}(-\gamma)].\\ &(z_1 < z, z' < \infty) \qquad (9)\end{aligned}$$

From (9), we can see that the $\hat{\boldsymbol{y}}\hat{\boldsymbol{y}}$-component of the electric Green dyads of the first kind derived here have the same form as (16) and (50a)–(51d) given by Collin [1] by letting $z_1=-\ell$, and (7) by Jarem [6] by letting $x_0=a/2$, $y_0=b/2$ and $z_1=-d$ while the rest are given for the first time, confirming the applicability of the present method.

B. Infinite Rectangular Waveguide

By simply substituting $z_1\to-\infty$ into (8), we see that the coefficient of the scattering Green dyad vanishes, i.e.

$$\overline{\boldsymbol{G}}_{\genfrac{}{}{0pt}{}{e1}{e2}s}(\boldsymbol{r},\boldsymbol{r}') = 0. \qquad (10)$$

[1] This relation can be easily proven when the waveguide is filled with the homogeneous lossy dielectric medium ($\gamma=\gamma_{Re}-j\gamma_{Im}$). If the medium is lossless, the Sommerfeld radiation condition must be used.

The result is expected since it agrees with our initial assumption that the electric Green dyads of the first and second kinds $\overline{\boldsymbol{G}}_{e{1 \atop 2}0}(\boldsymbol{r},\boldsymbol{r}')$ are those for an infinite rectangular waveguide. This particular case has been widely discussed by many researchers, e.g., Tai [2], Collin [1], Balanis [7], and Rahmat-Samii [4].

V. Conclusion

In short, this paper presents the electric dyadic Green's functions of the first and second kinds due to both the electric and equivalent magnetic current sources in rectangular cavities and waveguides. The dyadic Green's function of the second kind can be obtained usually from the Green dyad of the first kind by making the simple replacements $\boldsymbol{E} \rightarrow \boldsymbol{H}$, $\boldsymbol{H} \rightarrow -\boldsymbol{E}$, $\boldsymbol{J} \rightarrow \boldsymbol{M}$, $\boldsymbol{M} \rightarrow -\boldsymbol{J}$, $\mu \rightarrow \varepsilon$, and $\varepsilon \rightarrow \mu$. However, it is found in this paper that for a rectangular cavity or a rectangular waveguide, additional substitutions, i.e., even mode $(e) \rightarrow$ odd mode (o) and odd mode $(o) \rightarrow$ even mode (e), should be made in the derivations. The Green dyads of the second kind for an infinite and a semi-infinite rectangular waveguide and a rectangular cavity are provided in this paper. Besides, this paper shows that the dyadic Green's functions for the rectangular cavity can be simply reduced to those for the semi-infinite and infinite rectangular waveguides by letting the dimension $z_2 \rightarrow \infty$ and the dimensions $z_1 \rightarrow -\infty$, $z_2 \rightarrow \infty$, respectively. This is a very efficient and useful method for obtaining the Green dyads for rectangular waveguide with multiple loads after the Green dyads for a rectangular cavity filled with a multi-layered medium have been obtained. It should be emphasized that conversion of the current expression of dyadic Green's functions is preferably needed for the rapid convergence of the series summation [5], [6] while carrying out the numerical calculations.

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